Weak Power-Counting Theorem for the Renormalization of the Nonlinear Sigma Model in Four Dimensions

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The formulation of the nonlinear σ-model in terms of flat connection allows the construction of a perturbative solution of a local functional equation by means of cohomological techniques which are implemented in gauge theories. In this paper we discuss some properties of the solution at the one-loop level in $D = 4$. We prove the validity of a weak power-counting theorem in the following form: although the number of divergent amplitudes is infinite only a finite number of counterterms parameters have to be introduced in the effective action in order to make the theory finite at one loop, while respecting the functional equation (fully symmetric subtraction in the cohomological sense). The proof uses the linearized functional equation of which we provide the general solution in terms of local functionals. The counterterms are expressed in terms of linear combinations of these invariants and the coefficients are fixed by a finite number of divergent amplitudes. These latter amplitudes contain only insertions of the composite operators ϕ_0 (the constraint of the nonlinear σ-model) and F_μ (the flat connection). The structure of the functional equation suggests a hierarchy of the Green functions. In particular once the amplitudes for the composite operators ϕ_0 and F_μ are given all the others can be derived by functional derivatives. In this paper we show that at one loop the renormalization of the theory is achieved by the subtraction of divergences of the amplitudes at the top of the hierarchy. As an example we derive the counterterms for the four-point amplitudes.

KEY WORDS: renormalization; nonlinear sigma model.

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1. INTRODUCTION

Since a long time people realized that the nonlinear σ-model cannot be renormalized in a symmetric way by imposing global chiral symmetry already at one loop (Appelquist and Bernard, 1981; Ecker and Honerkamp, 1971; Tataru, 1975). Some of the unwanted (chiral breaking) terms can be disposed of by

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redefinition of the field (quartic divergences) (Charap, 1970; Gerstein *et al.*, 1971; Honerkamp and Meetz, 1971; Tataru, 1975). However some divergent terms of the one-loop off-shell pion-pion scattering amplitude still violate chiral symmetry and can be reabsorbed by redefinition of the field only if derivatives are allowed (Appelquist and Bernard, 1981). This strategy of removing the divergences never turned to a consistent program both for technical difficulty and for the impossibility of fixing the necessary finite subtractions. From these previous experiences it is clear that the renormalization of the nonlinear σ-model cannot be achieved by using chiral-invariant counterterms only. In particular one has to find a technique to implement the idea of field redefinition. This problem turns out to be closely related to the issue of identifying the good symmetry of the theory, i.e. the one that survives quantization.

We discuss here a unified solution (Ferrari, 2005) to both problems which makes use of a single scalar external source coupled to the constrained ϕ_0 field. The introduction of the composite operator ϕ_0 turns out to be unavoidable in order to discuss the implementation of chiral symmetry at the quantum level by means of the Ward-Takahashi identities.

Let us briefly outline the formalism by which we renormalize at one loop the nonlinear σ model. We consider (Ferrari, 2005) the scalar fields (ϕ_a) as parameters of a flat connection (gauge field with zero field strength). A local functional equation encoding the underlying local invariance property of the Haar measure in the path-integral

$$\delta\phi_a(x) = \frac{1}{2}\alpha_a(x)\phi_0(x) + \frac{g}{2}\epsilon_{abc}\phi_b(x)\alpha_c(x), \quad \delta\phi_0(x) = -\frac{g^2}{2}\alpha_a(x)\phi_a(x) \quad (1)$$

is then derived. Quantization is performed by imposing the functional equation on the 1-PI vertex functional in D-dimensions. The functional equation embodies the relevant symmetry of the full quantum theory .

The projection on the physical value $D \to 4$ requires a recursive subtraction procedure of the poles. The subtraction is implemented by a set of counterterms in the Feynman rules in such a way to respect the functional equation.

The counterterms are determined by exploiting a hierarchy inherent to the solutions of the functional equation. Since the counterterms have to be local functionals, the analysis of the functional equation can be limited to local solutions.

In this paper we provide a general classification of the local functionals which are solutions of the linearized equation. This is the relevant equation for the counterterms at one loop level. Moreover it is the equation which controls all possible finite subtractions in the dimensional renormalization scheme.

The invariants are integrated formal power series of local monomials in the pion fields ϕ_a, the external source $J_{a\mu}$ of the flat connection F_a^μ and the external source K_0 of the composite operator ϕ_0 (the constraint in the nonlinear σ-model). They can be classified by cohomological methods implemented in

gauge theories. This provides a useful insight into the underlying geometry of the quantum nonlinear σ model in $D = 4$.

The solution is governed by a weak power-counting theorem: although an infinite number of divergent amplitudes exists at one loop-level, only a finite number of them has to be evaluated in order to make the theory finite at one loop level while respecting the functional equation (fully symmetric subtraction in the cohomological sense). They correspond to amplitudes involving only the insertions of the composite operators ϕ_0 and F_a^μ, i.e. the amplitudes obtained by functional differentiation of the 1-PI vertex functional w.r.t. K_0's and J^μ's. These amplitudes are at the top of the hierarchy implied by the functional equation (ancestor amplitudes). They allow to fix uniquely the coefficients of the invariants entering in the solution which parameterizes the counterterms.

This is an extremely powerful tool for dealing with the intricacies of divergences of the nonlinear σ-model in $D = 4$, since all the other counterterms (i.e. those involving at least one ϕ field) can be derived from this solution by projection on the relevant monomials. We stress that when expanded on the basis of monomials in ϕ's and the external sources the solution contains an infinite number of terms, associated with the divergences of amplitudes with an arbitrarily high number of pion legs. All of them are needed in order to perform the one-loop renormalization of the model. It is a remarkable fact that they can be rewritten in terms of a finite number of invariants controlled by a finite number of independent coefficients.

As an example we obtain the counterterms for the set of four-point amplitudes. Moreover we apply the method to prove a simple criterion establishing the convergence of amplitudes which are divergent by naive power-counting but whose convergence is implied by the local functional equation.

This work is part of a program aiming to provide finite Feynman amplitudes at every order in the loop expansion of the nonlinear σ model in $D = 4$ in a symmetric scheme. The phenomenological implications of this subtraction strategy remain an open problem since at every order in $\hbar$ there is a new finite set of independent parameters associated to in principle admissible local counterterms. This aspect is shared by other approaches typically focused on the problem of giving a meaning to the loop corrections in chiral lagrangian models (Bijnens *et al.*, 2000; Gasser and Leutwyler, 1984, 1985).

The paper is organized as follows. In Section 2 we describe the subtraction procedure and the inherent weak power-counting theorem. In Section 3 we set up the cohomological framework needed to classify the local solutions of the linearized functional equation. The most general local solution is characterized in Section 4. Section 5 is devoted to the parameterization of the one-loop divergences in $D = 4$ in terms of local invariant solutions. As an application the counterterms of four-point amplitudes are derived in Section 6. In Section 7 we provide a comparison with similar results obtained in chiral lagrangian theories. Conclusions

are given in Section 8. Appendix A finally contains a derivation of the weak-power-counting formula.

2. SUBTRACTION PROCEDURE

In this section we deal with the nonlinear σ-model in the formulation given by the functional equation (Ferrari, 2005) which one derives from the local gauge transformations on the associated flat connection

$$F_\mu = \frac{i}{g}\Omega\partial_\mu\Omega^\dagger = \frac{1}{2}F_{a\mu}\tau^a,$$

$$\Omega = \frac{1}{m_D}(\phi_0 + ig\tau^a\phi_a), \quad \Omega^\dagger\Omega = 1, \quad \det\,\Omega = 1, \quad \phi_0^2 + g^2\phi_a^2 = m_D^2.$$
(2)

τ^a are the Pauli matrices and $m_D = m^{D/2-1}$. m is the mass scale of the theory.

The local transformations are

$$\Omega' = U\Omega,$$

$$F'_\mu = U F_\mu U^\dagger + \frac{i}{g}U\partial_\mu U^\dagger.$$
(3)

The local functional equation for the 1-PI generating functional follows from the standard path-integral formulation by using the classical action in D dimensions

$$\Gamma^{(0)} = \int d^D x \left(\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \frac{1}{2}g^2\frac{\phi_a\partial_\mu\phi_a\phi_b\partial^\mu\phi_b}{\phi_0^2} + K_0\phi_0 + J_{a\mu}F_a^\mu\right).$$
(4)

By exploiting the invariance of the Haar measure in the path-integral under the local gauge transformations one obtains

$$\left(\frac{m_D^2}{2}\partial^\mu\frac{\delta\Gamma}{\delta J_a^\mu} + g^2 K_0\phi_a + \frac{\delta\Gamma}{\delta K_0}\frac{\delta\Gamma}{\delta\phi_a} + g\epsilon_{abc}\frac{\delta\Gamma}{\delta\phi_b}\phi_c + 2D\left[\frac{\delta\Gamma}{\delta J}\right]_{ab}^\mu J_{b\mu}\right)(x) = 0$$
(5)

with

$$D[X]_{ab}^\mu = \partial^\mu\delta_{ab} - g\epsilon_{abc}X_c^\mu.$$
(6)

In order to construct the perturbative series we notice that $\Gamma^{(0)}$ in Eq. (4) is a solution to Eq. (5) and therefore we can read immediately from Eq. (4) the Feynman rules.

The 1-PI generating functional obtained from these rules is a solution to Eq. (5) in D dimensions. The projection of the D-dimensional solution on the physical value $D \to 4$ requires a recursive subtraction procedure.

The subtraction procedure follows the hierarchy implied by Eq. (5). This means that we fix at first the counterterms for the amplitudes involving only the composite operators F_a^μ and ϕ_0 (derivatives of Γ only w.r.t. $J_{a_1}^{\mu_1}, \ldots, J_{a_n}^{\mu_n}$, $\ldots, K_0, \ldots$). A simple dimensional analysis indicates that the removal of the poles in $D = 4$ has to be done on the Laurent expansion of the normalized amplitude

$$\left(\frac{m_D}{m}\right)^{2(n-1)} \Gamma_{J_{a_1}^{\mu_1} \ldots J_{a_n}^{\mu_n}}. \tag{7}$$

Equation (5) then constrains the correct factor for the amplitudes involving the fields ϕ_a and the composite operator ϕ_0.

We denote by $\Gamma_{\text{pol}}^{(n)}$ the corresponding pole part of the Laurent expansion of the n-th order vertex functional $\Gamma^{(n)}$.

Our conjecture is that order by order we can modify the Feynman rules by adding the counterterms required by dimensional subtraction, in such a way that Eq. (5) is satisfied (symmetric subtraction). At one loop level the removal of the pole part of the divergent amplitudes is by means of a solution of the linearized equation

$$\mathcal{S}_a\left(\Gamma_{\text{pol}}^{(1)}\right) = \left(\frac{m_D^2}{2} \partial^\mu \frac{\delta \Gamma_{\text{pol}}^{(1)}}{\delta J_a^\mu} - 2g\epsilon_{abc} \frac{\delta \Gamma_{\text{pol}}^{(1)}}{\delta J_{c\mu}} J_b^\mu + \frac{\delta \Gamma^{(0)}}{\delta K_0} \frac{\delta \Gamma_{\text{pol}}^{(1)}}{\delta \phi_a} + \frac{\delta \Gamma_{\text{pol}}^{(1)}}{\delta K_0} \frac{\delta \Gamma^{(0)}}{\delta \phi_a}\right.$$

$$\left. + g \, \epsilon_{abc} \frac{\delta \Gamma_{\text{pol}}^{(1)}}{\delta \phi_b} \phi_c\right)(x) = 0 \tag{8}$$

since at this order Eq. (5) coincides with the linearized Eq. (8).

The study of the solutions of Eq. (8) in terms of local functionals provides a necessary tool in order to make consistent the subtraction procedure outlined above. Their coefficients have to be chosen in such a way to remove the pole parts of the D-dimensional amplitudes.

As will be shown, these coefficients are uniquely fixed by the pole part of the divergent amplitudes which only involve the composite operators $F_{a\mu}$ and ϕ_0 (i.e. 1-PI Green functions obtained by differentiating Γ w.r.t. the sources $J_{a\mu}$ and K_0).

At each order n in the loop expansion only a finite number of them exists. There is indeed a weak power-counting for the external sources $J_{a\mu}$ and K_0. A n-loop graph with N_J insertions of the composite operator $F_{a\mu}$, N_{K_0} insertions of the composite operator ϕ_0 and no ϕ external legs is superficially convergent provided that

$$N_J + 2N_{K_0} > (D - 2)n + 2. \tag{9}$$

The derivation of the above formula is given in Appendix A. Equation (9) fixes the upper bound on the number of independent ancestor amplitudes.

The solutions of Eq. (8) will be given in terms of linear combinations of invariant local functionals. The coefficients of these invariants are in principle free parameters and they are constrained by the functional Eq. (5). The hierarchical structure of this equation might reduce drastically the number of independent divergent amplitudes to be evaluated. The simplest example of this is provided by the one-loop corrections where only the monomials in J and K_0 and their derivatives (present in the invariant solution) need to be computed in terms of the pole part of the amplitudes.

3. BACKGROUND FORMALISM

In order to classify the solutions to Eq. (8) it is convenient to introduce a set of local parameters $\omega_a(x)$ and rewrite Eq. (8) in the following equivalent form

$$
\delta\Gamma_{\text{pol}}^{(n)} \equiv \int d^4x \left(-\frac{m_D^2}{4} \partial^\mu \omega_a \frac{\delta\Gamma_{\text{pol}}^{(n)}}{\delta J_a^\mu} - g\epsilon_{abc}\omega_a J_b^\mu \frac{\delta\Gamma_{\text{pol}}^{(n)}}{\delta J_{c\mu}} \right.
$$
$$
\left. + \left(\frac{\omega_a}{2} \frac{\delta\Gamma^{(0)}}{\delta K_0} + \frac{g}{2}\epsilon_{abc}\phi_b\omega_c \right) \frac{\delta\Gamma_{\text{pol}}^{(n)}}{\delta\phi_a} + \frac{\omega_a}{2} \frac{\delta\Gamma^{(0)}}{\delta\phi_a} \frac{\delta\Gamma_{\text{pol}}^{(n)}}{\delta K_0} \right) = 0 \quad (10)
$$

The geometrical meaning of the above equation becomes clear after the rescaling

$$
\tilde{J}_{a\mu} = -\frac{4}{m_D^2} J_{a\mu}. \tag{11}
$$

$\tilde{J}_{a\mu}$ transforms as a (background) gauge connection under the action of δ while $\Omega = \frac{1}{m_D}(\phi_0 + ig\tau^a\phi_a)$ transforms in the fundamental representation. For later use we notice that the transformation of K_0 is proportional to the classical equation of motion for ϕ_a.

There is a BRST differential s (Becchi *et al.*, 1975, 1976; Piguet and Sorella, 1995) associated with the transformation in Eq. (10). It is obtained by promoting the parameters ω_a to classical local anticommuting parameters. Global chiral symmetry has been discussed in a similar fashion with the use of constant ghosts in Blasi and Collina (1987). The action of s on $\tilde{J}_{a\mu}$, ϕ_a and K_0 is induced by the action of δ, i.e.

$$
s\tilde{J}_{a\mu} = \partial_\mu\omega_a + g\epsilon_{abc}\tilde{J}_{b\mu}\omega_c, \quad s\phi_a = \frac{1}{2}\omega_a\phi_0 + \frac{1}{2}g\epsilon_{abc}\phi_b\omega_c,
$$
$$
sK_0 = \frac{1}{2}\omega_a \frac{\delta\Gamma^{(0)}}{\delta\phi_a}. \tag{12}
$$

The operator s becomes nilpotent provided that we extend its action to ω_a by setting

$$s\omega_a = -\frac{1}{2}g\epsilon_{abc}\omega_b\omega_c. \tag{13}$$

A conserved Faddeev-Popov ($\Phi\Pi$) charge can be introduced by requiring that all variables with the exception of ω_a are $\Phi\Pi$-neutral and $\Phi\Pi(\omega_a) = 1$.

Equation (10) is equivalent to

$$s\Gamma^{(n)}_{\text{pol}} = 0 \tag{14}$$

since there are no variables with negative $\Phi\Pi$-charge (thus forbidding s-exact solutions $Y^{(n)} = s X^{(n)}$, where $X^{(n)}$ has $\Phi\Pi$-charge -1, which automatically fulfill $s Y^{(n)} = 0$ by the nilpotency of s).

The advantage of the BRST formulation of the local functional equation provided by Eq. (14) is that it allows to make use of the cohomological techniques implemented in gauge theories (Barnich *et al*., 2000; Gomis *et al*., 1995; Henneaux and Wilch, 1998; Piguet and Sorella, 1995) in order to derive an exhaustive classification of the solutions.

4. SOLUTIONS OF THE LINEARIZED FUNCTIONAL EQUATION

We now move to the study of Eq. (14). The recursive subtraction of the poles is implemented by a set of counterterms in the Feynman rules. It is required that they are local functionals solution of Eq. (14).

For renormalizable theories the power-counting theorem puts dimensionality bounds on them and so this limits the number of independent monomials. For non-renormalizable theories as the one we are dealing with this constraint on the number is no more present.

On general grounds the required counterterms might in some cases reduce to a polynomial if the perturbative expansion is cut to a finite loop order. We will show that this is not the case for the nonlinear σ-model even at one loop level: there exist divergent amplitudes involving any number of ϕ's.

This apparently wild behavior is tamed by an extremely powerful hierarchy when Eq. (14) is used in order to parameterize the one-loop divergences. Indeed it turns out that the counterterms are controlled by a linear combination of a finite number of invariants which are solutions to Eq. (14), as a consequence of the weak power-counting on K_0 and $J_{a\mu}$. Once the relevant linear combination is known, all the divergences for amplitudes involving any number of ϕ's and external sources are obtained by projection on the relevant monomial in ϕ's, K_0 and $J_{a\mu}$. Equation (14) thus provides an extremely powerful and efficient tool for the classification of the UV divergences in the model at hand.

In order to exploit Eq. (14) we first need to find the most general solution to Eq. (14) in the space of integrated local functionals (in the sense of local formal power series) spanned by ϕ_a, K_0, $J_{a\mu}$ and their derivatives. This amounts to characterize the cohomology of the nilpotent differential s in Eq. (12) in the sector of $\Phi\Pi$-neutral local functionals.

The required solution can be found rather easily by noticing that the following combination

$$\overline{K}_0 = \frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a} \tag{15}$$

is s-invariant. In the above equation we have set

$$S_0 = \frac{m_D^2}{8} \int d^D x \left(F_{a\mu} + \frac{4}{m_D^2} J_{a\mu} \right)^2. \tag{16}$$

By exploiting the invariance of S_0 under s we obtain

$$
\begin{aligned}
s\overline{K}_0 &= \frac{m_D^2}{2\phi_0} \omega_a \frac{\delta \Gamma^{(0)}}{\delta \phi_a} + \frac{g^2 m_D^2}{2\phi_0^2} K_0 \, \omega_a \phi_a + \left[s, -\phi_a \frac{\delta}{\delta \phi_a} \right] S_0 \\
&= \frac{m_D^2}{2\phi_0} \omega_a \frac{\delta S_0}{\delta \phi_a} - \frac{g^2 m_D^2}{2\phi_0^2} K_0 \, \omega_a \phi_a + \frac{g^2 m_D^2}{2\phi_0^2} K_0 \, \omega_a \phi_a \\
&\quad + \left[s, -\phi_a \frac{\delta}{\delta \phi_a} \right] S_0 \\
&= \frac{m_D^2}{2\phi_0} \omega_a \frac{\delta S_0}{\delta \phi_a} + \left[s, -\phi_a \frac{\delta}{\delta \phi_a} \right] S_0.
\end{aligned}
\tag{17}
$$

By taking into account that S_0 does not depend on K_0 we also get

$$\left[s, -\phi_a \frac{\delta}{\delta \phi_a} \right] S_0 = -\frac{1}{2} \omega_a \phi_0 \frac{\delta S_0}{\delta \phi_a} - \frac{g^2}{2\phi_0} \omega_a \phi_b^2 \frac{\delta S_0}{\delta \phi_a}. \tag{18}$$

Use of Eq. (18) into Eq. (17) yields finally

$$
\begin{aligned}
s\overline{K}_0 &= \frac{1}{2\phi_0} \omega_a \left(m_D^2 - g^2 \phi_b^2 \right) \frac{\delta S_0}{\delta \phi_a} - \frac{1}{2} \omega_a \phi_0 \frac{\delta S_0}{\delta \phi_a} \\
&= \frac{1}{2} \omega_a \phi_0 \frac{\delta S_0}{\delta \phi_a} - \frac{1}{2} \omega_a \phi_0 \frac{\delta S_0}{\delta \phi_a} = 0
\end{aligned}
\tag{19}
$$

where use has been made of the last of Eqs. (2).

Since the transformation in Eq. (15) is invertible we can change variables and use ϕ_a, $J_{a\mu}$ and $\overline{K}_0$. $\overline{K}_0$ is invariant under s while the s-variation of ϕ_a and $J_{a\mu}$ does not contain $\overline{K}_0$.

Hence the computation of the cohomology of s in Eq. (12) in the $\Phi\Pi$-neutral sector reduces to that of the BRST differential for the gauge group $SU(2)$

(non-linearly represented on the group element Ω) in the space of local functionals with zero $\Phi\Pi$-charge. This is easily seen by identifying the $SU(2)$ connection with $\tilde{J}_{a\mu}$ in Eq. (11), while ϕ_a are the parameters controlling the non-linear representation of the gauge group by the matrix Ω. $\overline{K}_0$ is an additional variable which does not transform under s.

The cohomology of the BRST differential for non-linear representations of the gauge group $SU(2)$ is known in full generality (Barnich *et al.*, 2000; Henneaux and Wilch, 1998). This allows us to state the following

Proposition. The most general local solution to Eq. (14) is an integrated BRST (Eq. (12))-invariant local formal power series constructed from the invariant combination $\overline{K}_0$ and its ordinary derivatives, the undifferentiated group element Ω and the combination $F_a^\mu + \frac{4}{m_D^2} J_a^\mu$ and its subsequent covariant derivatives w.r.t. F_μ.

The proof of this result is based on cohomological techniques and is detailed in Barnich *et al.* (2000), Henneaux and Wilch (1998). Here we only wish to make a few comments.

The combination

$$F_a^\mu + \frac{4}{m_D^2} J_a^\mu = F_a^\mu - \tilde{J}_a^\mu \tag{20}$$

is the difference of two $SU(2)$ connections and thus it transforms in the adjoint representation of $SU(2)$:

$$s\left(F_a^\mu - \tilde{J}_a^\mu\right) = g\epsilon_{abc}\left(F_b^\mu - \tilde{J}_b^\mu\right)\omega_c. \tag{21}$$

Moreover we notice that covariant derivatives have to be understood only w.r.t. F_μ. Covariant derivatives w.r.t. $\tilde{J}^\mu$ can also be used in order to construct invariants. However these invariants are not independent, since a covariant derivative w.r.t. $\tilde{J}^\mu$ can be replaced by a covariant derivative w.r.t. F^μ plus a term containing the combination $F^\mu + \frac{4}{m_D^2} J^\mu$.

Finally in the sector with at least one derivative there is still the freedom to perform an integration by parts in order to reduce the number of independent invariants. Once this ambiguitiy is taken into account one gets the set of independent invariants on which to project the solutions to Eq. (14).

The above Proposition is a very powerful result allowing for a simple constructive characterization of the solutions to Eq. (14). In the next section we will show how to make use of it in order to specify completely the whole set of one-loop counterterms.

5. ONE-LOOP COUNTERTERMS

From the above discussion we can deal with the one-loop corrections in $D = 4$ by writing the most general local solution to Eq. (14) compatible with the

weak power-counting. Since Eq. (14) is linear, the solution is a linear combination of the following invariants (all covariant derivatives are understood w.r.t. the flat connection F_μ):

$$\mathcal{I}_1 = \int d^D x \left[D_\mu \left(F + \frac{4}{m_D^2} J \right) \right]_{v\,a} \left[D^\mu \left(F + \frac{4}{m_D^2} J \right)^v \right]_a ,$$

$$\mathcal{I}_2 = \int d^D x \left[D_\mu \left(F + \frac{4}{m_D^2} J \right)^\mu \right]_a \left[D_v \left(F + \frac{4}{m_D^2} J \right)^v \right]_a ,$$

$$\mathcal{I}_3 = \int d^D x \, \epsilon_{abc} \left[D_\mu \left(F + \frac{4}{m_D^2} J \right)_v \right]_a \left(F + \frac{4}{m_D^2} J \right)_b^\mu \left(F + \frac{4}{m_D^2} J \right)_c^v ,$$

$$\mathcal{I}_4 = \int d^D x \left(\frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a} \right)^2 ,$$

$$\mathcal{I}_5 = \int d^D x \left(\frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a} \right) \left(F + \frac{4}{m_D^2} J \right)^2 ,$$

$$\mathcal{I}_6 = \int d^D x \left(F + \frac{4}{m_D^2} J \right)^2 \left(F + \frac{4}{m_D^2} J \right)^2 ,$$

$$\mathcal{I}_7 = \int d^D x \left(F + \frac{4}{m_D^2} J \right)_a^\mu \left(F + \frac{4}{m_D^2} J \right)_a^v$$
$$\left(F + \frac{4}{m_D^2} J \right)_{b\mu} \left(F + \frac{4}{m_D^2} J \right)_{bv} . \tag{22}$$

A few comments on this list are in order: $\mathcal{I}_1$ and $\mathcal{I}_2$ describe the pole part of the 2-point function $\Gamma_{JJ}^{(1)}$. $\mathcal{I}_3$ is the only invariant that can yield the counterterm associated with $\Gamma_{JJJ}^{(1)}$. Finally $\mathcal{I}_6$ and $\mathcal{I}_7$ control the pole part of the 4-point function $\Gamma_{JJJJ}^{(1)}$, while the 2-point function $\Gamma_{K_0 K_0}^{(1)}$ and the 3-point function $\Gamma_{K_0 JJ}^{(1)}$ are related to $\mathcal{I}_4$ and $\mathcal{I}_5$. We notice that the functional equation in Eq. (5) allows to derive $\Gamma_{K_0 K_0}^{(1)}$ and $\Gamma_{K_0 JJ}^{(1)}$ from $\Gamma_{JJJJ}^{(1)}$, $\Gamma_{JJJ}^{(1)}$ and $\Gamma_{JJ}^{(1)}$. Therefore only three amplitudes have to be computed.

The correct linear combination of the invariants has to be found by comparison with the solution of Eq. (5) which is valid in D-dimensions. Therefore the coefficients must contain the correct power of m_D. Once these coefficients have been established all the one-loop divergences for amplitudes involving any number of ϕ's are described by the projection of the solution on the relevant monomial. In fact all the amplitudes involving at least one ϕ field can be derived by subsequent use of the functional Eq. (5).

We denote by $\hat{\Gamma}^{(1)} = -\Gamma^{(1)}_{pol}$ the one-loop divergent counterterms.

By direct computation one finds $\hat{\Gamma}^{(1)}[JJ]$ and $\hat{\Gamma}^{(1)}[JJJ]$ (Ferrari, 2005)

$$\hat{\Gamma}^{(1)}[JJ] = \frac{1}{D-4} \left(\frac{m}{m_D}\right)^2 \frac{g^2}{12\pi^2 m^4} \int d^D x \, J_a^\mu (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) J_a^\nu,$$

$$\hat{\Gamma}^{(1)}[JJJ] = \frac{1}{D-4} \frac{1}{3\pi^2} \left(\frac{g}{m^2}\right)^3 \left(\frac{m}{m_D}\right)^4 \int d^D x \, \epsilon_{abc} \partial_\mu J_{av} J_b^\mu J_c^\nu. \quad (23)$$

This fixes the coefficients of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ which enter into the solution in the combination

$$-\frac{1}{D-4} \frac{1}{12} \frac{g^2}{(4\pi)^2} \frac{m_D^2}{m^2} (\mathcal{I}_1 - \mathcal{I}_2 - g\mathcal{I}_3). \quad (24)$$

Direct computation of the pole part of $\Gamma^{(1)}_{JJJJ}$ gives

$$\hat{\Gamma}^{(1)}[JJJJ] = \frac{1}{D-4} \frac{1}{3(4\pi)^2} \left(\frac{2g}{m^2}\right)^4 \left(\frac{m}{m_D}\right)^6$$

$$\int d^D x \left(J_{a\mu} J_a^\mu J_{b\nu} J_b^\nu + 2 J_{a\mu} J_{a\nu} J_b^\mu J_b^\nu\right). \quad (25)$$

This in turn fixes the coefficients of $\mathcal{I}_6$ and $\mathcal{I}_7$ in the combination

$$\frac{1}{D-4} \frac{1}{(4\pi)^2} \frac{g^4}{48} \frac{m_D^2}{m^2} \left(\mathcal{I}_6 + 2\mathcal{I}_7\right). \quad (26)$$

Finally from the counterterms

$$\hat{\Gamma}^{(1)}[K_0 K_0] = \frac{1}{D-4} \frac{3g^4}{2m^2} \frac{1}{(4\pi)^2} \int d^D x \, K_0^2(x) \quad (27)$$

and

$$\hat{\Gamma}^{(1)}[K_0 JJ] = \frac{1}{D-4} \frac{8g^4}{m^5} \frac{1}{(4\pi)^2} \left(\frac{m}{m_D}\right)^3 \int d^D x \, K_0(x) J^2(x) \quad (28)$$

we get the coefficients of $\mathcal{I}_4$ and $\mathcal{I}_5$:

$$\frac{1}{D-4} \frac{1}{(4\pi)^2} \frac{3}{2} \frac{g^4}{m^2 m_D^2} \mathcal{I}_4 + \frac{1}{D-4} \frac{1}{(4\pi)^2} \frac{1}{2} \frac{g^4}{m^2} \mathcal{I}_5. \quad (29)$$

Therefore the full set of one-loop divergent counterterms is given by the functional

$$\hat{\Gamma}^{(1)} = \frac{1}{D-4} \left[-\frac{1}{12} \frac{g^2}{(4\pi)^2} \frac{m_D^2}{m^2} (\mathcal{I}_1 - \mathcal{I}_2 - g\mathcal{I}_3) + \frac{1}{(4\pi)^2} \frac{g^4}{48} \frac{m_D^2}{m^2} (\mathcal{I}_6 + 2\mathcal{I}_7) \right.$$

$$+ \frac{1}{(4\pi)^2} \frac{3}{2} \frac{g^4}{m^2 m_D^2} \mathcal{I}_4 + \frac{1}{(4\pi)^2} \frac{1}{2} \frac{g^4}{m^2} \mathcal{I}_5 \Bigg].\tag{30}$$

These are the counterterms to be used in D-dimensional perturbation theory. This is the reason why m_D is put in evidence. Moreover the presence of m_D both in the coefficients and the invariants fixes non-trivial finite parts of the counterterms beyond the pole part in $\frac{1}{D-4}$. These finite parts are non-trivial since they are needed to maintain the validity of the functional equation after subtraction.

Equation (30) is not the most general solution. One can always add finite solutions of $sX = 0$. It is a choice that we make in this paper to perform a minimal subtraction on the basis of simplicity and elegance.

The explicit form of the counterterms (30) allows us to comment on two further important points.

One is the issue of chiral invariance of the counterterms at one loop. By direct inspection one sees that, after putting $J_\mu^a = K_0 = 0$, $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_6$ and $\mathcal{I}_7$ are chiral invariant (global transformation) while both $\mathcal{I}_4$ and $\mathcal{I}_5$ are not chiral invariant. Therefore the counterterms at one loop do not maintain chiral invariance as noted in Appelquist and Bernard (1981), Ecker and Honerkamp (1971), and Tataru (1975).

As a last point Eq. (30) accounts for the fact that the chiral-breaking counterterms are associated to the renormalization of the insertion of the composite operator ϕ_0 coupled to the source K_0.

6. EXAMPLES

The use of Eq. (30) is straightforward. One needs only to perform the relevant functional derivatives of the local functional $\hat{\Gamma}^{(1)}$.

As an example we can get the counterterm for the four-point function by projecting $\hat{\Gamma}^{(1)}$ in Eq. (30) on the monomials involving ϕ_a, K_0 and $J_{a\mu}$. First we consider the four-point function of the scalar fields. By direct computation the projection of the combination $\mathcal{I}_1 - \mathcal{I}_2 - g\mathcal{I}_3$ on the relevant monomials is zero, while the contribution from $\mathcal{I}_6 + 2\mathcal{I}_7$ and $\mathcal{I}_4, \mathcal{I}_5$ gives rise to

$$\hat{\Gamma}^{(1)}[\phi\phi\phi\phi] = -\frac{1}{D-4} \frac{g^4}{m_D^2 m^2 (4\pi)^2}$$

$$\int d^D x \Bigg[-\frac{1}{3} \partial_\mu \phi_a \partial^\mu \phi_a \partial_\nu \phi_b \partial^\nu \phi_b - \frac{2}{3} \partial_\mu \phi_a \partial_\nu \phi_a \partial^\mu \phi_b \partial^\nu \phi_b$$

$$-\frac{3}{2} \phi_a \Box \phi_a \phi_b \Box \phi_b - 2\phi_a \Box \phi_a \partial_\mu \phi_b \partial^\mu \phi_b \Bigg].\tag{31}$$

The terms in the first line between square brackets are associated to global chiral-invariant counterterms (Appelquist and Bernard, 1981; Faddeev and Slavnov,

1973). They are generated by the combination $\mathcal{I}_6 + 2\mathcal{I}_7$. These invariants are constructed from the geometrical quantities given by the flat connection F_μ and the background connection $\tilde{J}_\mu$. The terms in the second line are obtained from the projection of the invariants $\mathcal{I}_4$ and $\mathcal{I}_5$, which are controlled by $\hat{\Gamma}^{(1)}_{K_0 K_0}$ and $\hat{\Gamma}^{(1)}_{K_0 JJ}$. The latter encode the renormalization of the external source K_0. In Appelquist and Bernard (1981), Tataru (1975) they were obtained by means of a (non-locally invertible) field redefinition of ϕ_a.

We also provide the counterterms for the remaining four-point functions. By projection on the relevant monomials we obtain

$$\hat{\Gamma}^{(1)}[JJJ\phi] = \frac{1}{D-4} \frac{8}{(4\pi)^2} \frac{g^4}{m^2} \frac{1}{m_D^5}$$

$$\int d^D x \, \phi_a \left(2\partial J^a J^2 - \frac{8}{3} \partial_\nu \left(J_a^\nu J^2 \right) - \frac{4}{3} \partial_\nu \left(J_c^\nu J_c^\mu J_{a\mu} \right) \right) \quad (32)$$

$$\hat{\Gamma}^{(1)}[JJ\phi\phi] = -\frac{1}{D-4} \frac{4}{(4\pi)^2} \frac{g^4}{m^2} \frac{1}{m_D^4} \int d^D x \, \left(\frac{4}{3} \partial_\mu \phi_a \partial^\mu \phi_a J^2 + \partial_\mu \phi_a^2 \partial^\mu J^2 \right.$$

$$+ \frac{1}{2} \phi_c \phi_b \partial J_b \partial J_c - \frac{4}{3} \partial^\mu \phi_c \partial_\mu \phi_b J_c^\nu J_{b\nu}$$

$$+ \frac{8}{3} \phi_c \partial_\mu \phi_b \left(J_c^\nu \partial_\nu J_b^\mu - J_b^\nu \partial_\nu J_c^\mu \right) + \frac{1}{2} \phi_c \phi_b J_c^\nu \partial_\nu \partial J_b$$

$$- \frac{2}{3} \partial^\mu \phi_c \phi_b \left(\partial_\mu J_c^\nu J_{b\nu} - J_c^\nu \partial_\mu J_{b\nu} \right)$$

$$\left. - \frac{8}{3} \partial_\mu \phi_c \partial_\nu \phi_b J_c^\nu J_b^\mu - \frac{4}{3} \partial_\mu \phi_c \partial_\nu \phi_c J_b^\nu J_b^\mu \right) \quad (33)$$

$$\hat{\Gamma}^{(1)}[J\phi\phi\phi] = \frac{1}{D-4} \frac{2g^4}{m^2(4\pi)^2} \frac{1}{m_D^3} \int d^D x$$

$$\left(\frac{1}{2} J_a^\mu \partial_\mu \phi_a \Box \phi^2 - J_a^\mu \partial_\mu \phi_a \partial_\nu \phi_d \partial^\nu \phi_d \right.$$

$$+ J_a^\mu \phi_a \partial_\mu (\partial_\nu \phi_d \partial^\nu \phi_d) - \frac{3}{2} J_a^\mu \phi_a \partial_\mu \Box \phi^2$$

$$\left. - \frac{2}{3} J_a^\mu \phi_c (\Box g_{\mu\nu} - \partial_\mu \partial_\nu)(\partial^\nu \phi_c \phi_a) + 2 J_a^\mu \partial_\mu \phi_c \partial^\nu \phi_a \partial_\nu \phi_c \right) \quad (34)$$

$$\hat{\Gamma}^{(1)}[K_0 K_0 \phi\phi] = \frac{1}{D-4} \frac{1}{(4\pi)^2} \frac{3}{2} \frac{g^6}{m^2 m_D^2} \int d^D x \, K_0^2 \phi_a^2 \quad (35)$$

$$\hat{\Gamma}^{(1)}[K_0 J \phi \phi] = -\frac{1}{D-4}\frac{1}{(4\pi)^2}\frac{4g^5}{m^2 m_D^3}\int d^D x\, K_0 \epsilon_{abc}\partial_\mu \phi_b \phi_c J_a^\mu. \tag{36}$$

We would like to make some additional comments on Eq. (30). First we notice that the expansion of $\hat{\Gamma}^{(1)}$ on a basis of monomials in ϕ, K_0, J_μ and their derivatives contains terms of arbitrarily high order in the number of ϕ's. Therefore there is an infinite set of divergent amplitudes involving the fields ϕ. Nevertheless they are all controlled by Eq. (30), which contains only a finite number of invariants.

Equation (30) provides a full control on the divergences of the theory. For instance the amplitude $\Gamma^{(1)}_{JJJJ\phi}$ is divergent by simple power-counting. It is convergent due to the cancellations implied by the functional equation in Eq. (5), as it can be explicitly checked. This can be seen in an easier way from Eq. (30) by noticing that the projection of $\hat{\Gamma}^{(1)}$ on $JJJJ\phi$ is zero.

More generally the following simple criterion holds true: whenever the projection of $\hat{\Gamma}^{(1)}$ on some monomial is zero, the corresponding amplitude is finite.

7. COMPARISON WITH CHIRAL LAGRANGIAN THEORIES

In the present work we focused on the symmetric subtraction of the divergences in the nonlinear sigma model and therefore particular care has been put to write the most general counterterms in D-dimensions (addressing in particular their dependence on m_D). Moreover the powerful strategy, based on the hierarchy of the functional equation, plays a crucial role for the validity of the weak power-counting.

The counterterms obtained in Eq. (30) can be compared with a similar result in chiral lagrangian models. In order to make the comparison an easy task we use in this Section a set of notations very close to the ones adopted in the specialized literature on chiral perturbation theory.

The counterterms of the chiral lagrangian will be written in terms of the invariants $\mathcal{I}_1 - \mathcal{I}_7$ by means of two quantities that are essential in our approach: the external currents ξ^i coupled to the fields U^i are introduced as a Legendre conjugate

$$\xi^i = -\frac{\delta \Gamma^{(0)}}{\delta U^i}, \tag{37}$$

and moreover the flat connection is introduced by

$$F_\mu = i U \partial_\mu U^\dagger = F_\mu^i \frac{\tau^i}{2} \tag{38}$$

where

$$U = U_0 + i U^i \tau^i. \tag{39}$$

The tree-level effective action is

$$\Gamma^{(0)} = \int d^4x \left(\frac{f^2}{4} \text{Tr}(F_\mu - L_\mu)^2 + \xi^0 U^0 \right). \tag{40}$$

In this notations the s operator becomes (in the zero ghost number sector)

$$s = \int d^4x \frac{\omega^a}{2} \left((\delta^{ab} U^0 + \epsilon^{abc} U^c) \frac{\delta}{\delta U^b} + \frac{\delta \Gamma^{(0)}}{\delta U^a} \frac{\delta}{\delta \xi^0} \right.$$

$$\left. + (-2\partial^\mu \delta^{ab} + 2\epsilon^{abc} L^c_\mu) \frac{\delta}{\delta L^b_\mu} \right). \tag{41}$$

One gets

$$s F^i_\mu = \partial_\mu \omega^i + \epsilon^{ijk} F^j_\mu \omega^k,$$

$$s L^i_\mu = \partial_\mu \omega^i + \epsilon^{ijk} L^j_\mu \omega^k. \tag{42}$$

It is straightforward to find the transformation properties of ξ^i:

$$s\xi^i = s \left(-\frac{\delta \Gamma^{(0)}}{\delta U^i} \right) = -\left[s, \frac{\delta}{\delta U^i} \right] \Gamma^{(0)} - \frac{\delta}{\delta U^i} (s\Gamma^{(0)})$$

$$= +\frac{\omega^i}{2} \xi^0 - \epsilon^{iab} \omega^a \xi^b. \tag{43}$$

Moreover

$$s\xi^0 = \frac{\omega^a}{2} \frac{\delta \Gamma^{(0)}}{\delta U^a} = -\frac{1}{2} \omega^a \xi^a. \tag{44}$$

Therefore (ξ^0, ξ^i) transform like (U^0, U^i). The transformation properties in Eqs. (42), (43) and (44) allow the conctruction of invariant local counterterms by using the covariant derivatives

$$\nabla_\mu U \equiv (\partial_\mu - i L_\mu) U = i(F - L)_\mu U. \tag{45}$$

An useful relation can be obtained from the identity

$$s \int d^4x \text{Tr}\,(F - L)^2 = 0 \tag{46}$$

i.e.

$$\frac{1}{2} (\delta^{ab} U^0 + \epsilon^{abc} U^c) \frac{\delta}{\delta U^b} \int d^4x \text{Tr}\,(F - L)^2 = -2(\partial^\mu \delta^{ab} - \epsilon^{abc} L^c_\mu)(F - L)^\mu_b$$

$$= -2D[L]_{ab\mu}(F - L)^\mu_b. \tag{47}$$

The square is

$$(\delta^{bb'} - U^b U^{b'})\frac{\delta}{\delta U^b}\int d^4x \operatorname{Tr}(F-L)^2 \frac{\delta}{\delta U^{b'}}\int d^4y \operatorname{Tr}(F-L)^2$$
$$= 16\left(D[L]_{ab\mu}(F-L)^\mu_b\right)^2. \tag{48}$$

By using Eq. (48) one gets

$$\int d^4x(\xi^0\xi^0 + \vec{\xi}^2) = \int d^4x \left(\xi^0\xi^0 + \left[-\frac{\delta}{\delta U^b}\int d^4y \frac{f^2}{4}\operatorname{Tr}(F-L)^2 + \xi^0\frac{U^b}{U^0}\right]^2\right)$$
$$= \int d^4x \left(\left(\frac{\xi^0}{U^0} - U^b\frac{\delta}{\delta U^b}\int d^4y \frac{f^2}{4}\operatorname{Tr}(F-L)^2\right)^2 + \frac{1}{4}\left(D[L]_{ab\mu}(F-L)^\mu_b\right)^2\right), \tag{49}$$

where the last two terms can be identified as $\mathcal{I}_4$ and $\mathcal{I}_2$ in Eq. (22).

The correspondence with our conventions is obtained by setting

$$f = m_D, \quad g = 1, \quad U^0 = \frac{1}{m_D}\phi_0, \quad U^i = \frac{1}{m_D}\phi_i,$$
$$\xi^0 = m_D K_0, \quad \tilde{J}^\mu_i = L^{\mu i}. \tag{50}$$

The correspondence with the notations used in Gasser and Leutwyler (1984) is obtained by the following prescription

$$f = F, \quad \xi^0 = F^2\chi^0, \quad \tilde{\chi} = 0, \quad L^i_\mu = \left(a^i_\mu + v^i_\mu\right), \quad a^i_\mu = v^i_\mu. \tag{51}$$

By using Eqs. (50) and (51) we are in a position to express the chiral invariants of Gasser and Leutwyler (1984) on the basis given by the invariants $\mathcal{I}_1 - \mathcal{I}_7$:

$$\int d^4x\, (\nabla^\mu U^T \nabla_\mu U)^2 = \frac{1}{16}\mathcal{I}_6,$$

$$\int d^4x\, (\nabla^\mu U^T \nabla^\nu U)(\nabla_\mu U^T \nabla_\nu U) = \frac{1}{16}\mathcal{I}_7,$$

$$\int d^4x\, (\chi^T U)^2 = \frac{1}{m^4}\mathcal{I}_4,$$

$$\int d^4x\, (\nabla^\mu \chi^T \nabla_\mu U) = \frac{1}{4m^2}\mathcal{I}_5 - \frac{1}{4}\mathcal{I}_2,$$

$$\int d^4x\, (U^T F^{\mu\nu} F_{\mu\nu} U) = -\frac{1}{2}\mathcal{I}_1 + \frac{1}{2}\mathcal{I}_2 + \mathcal{I}_3 - \frac{1}{4}\mathcal{I}_6 + \frac{1}{4}\mathcal{I}_7,$$

$$\int d^4x\, (\nabla^\mu U)^T F_{\mu\nu}(\nabla^\nu U) = \frac{1}{4}\mathcal{I}_3 + \frac{1}{8}(\mathcal{I}_7 - \mathcal{I}_6),$$

$$\int d^4x\, (\chi^T \chi) = \frac{1}{m^4}\mathcal{I}_4 + \frac{1}{4}\mathcal{I}_2,$$

$$\int d^4x \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} = -2\mathcal{I}_1 + 2\mathcal{I}_2 + 4\mathcal{I}_3 - \mathcal{I}_6 + \mathcal{I}_7. \tag{52}$$

By making use of the above correspondence table it is then easy to verify that the divergent part of the counterterms obtained in Gasser and Leutwyler (1984) coincide with those given by Eq. (30).

One should however realize that the D-dimensional counterterms in Eq. (30) have a non-trivial dependence on m_D. The latter gives rise to finite parts which are crucial in order to maintain the validity of the functional equation in the recursive subtraction procedure at higher orders in the loop expansion. See for instance the explicit calculation at the two-loop level in Ferrari and Quadri (2006).

8. CONCLUSIONS

In this paper we have shown that at the one loop level the nonlinear σ-model can be renormalized by using dimensional subtraction in such a way that the defining functional equation is preserved.

The construction of the counterterms is based on the symmetry property generated by a nilpotent operator s which transforms fields and external sources in a BRST fashion. This operator is obtained as a linearized form of the functional equation in the loop expansion.

Both the functional equation and the operator s express a hierarchy structure of the Green functions. The ancestors at the top are given by the Green functions involving only the external sources of the flat connection F_μ and the constrained field ϕ_0.

A weak power-counting theorem then follows stating that, although the number of divergent amplitudes is infinite, only a finite number of counterterms parameters has to be introduced in the effective action in order to make the theory finite at one loop level while respecting the functional equation (fully symmetric subtraction in the cohomological sense).

The counterterms are then a linear combination of the s-invariants. The weak power-counting limits the number of invariants needed for the complete renormalization at the one-loop level. The amplitudes involving only insertions of the composite operators F_a^μ and ϕ_0 uniquely fix the coefficients of the local invariants entering in the linear combination which parameterizes the one-loop counterterms. All the remaining divergent amplitudes can be obtained by projection of the linear combination on the appropriate monomials.

The structure of the counterterms reveals that both the pole parts and the finite parts have to be properly fixed in order to maintain the validity of the unsubtracted functional equation. Moreover by inspection one sees that some

of the counterterms are not chiral invariant. These are associated to invariants containing the external source of the constrained field ϕ_0.

As an example we have derived the expressions for the counterterms of the set of four-point functions. Amplitudes associated with monomials which are not contained in this linear combination are convergent (although their superficial degree of divergence may be non-negative).

In $D = 4$ the whole structure of one-loop divergences of the nonlinear σ-model is determined in terms of the finite set of invariants with given coefficients in Eq. (30). This allows to renormalize completely the theory at one-loop order.

We emphasize that the D-dimensional counterterms in Eq. (30) contain a non-trivial dependence on m_D. The latter gives rise to finite parts which prove to be crucial in order to maintain the validity of the functional equation in the recursive subtraction procedure at higher orders in the loop expansion.

A WEAK POWER-COUNTING FOR J_μ AND K_0

Let G be a n-loop graph with I internal lines and a certain set of vertices described by a collection of non-negative integers

$$
\begin{aligned}
&\left\{ V_J^{(2)}, V_J^{(3)}, V_J^{(5)}, \ldots, V_J^{(2p+1)}, \ldots, \right. \\
&\ V_{K_0}^{(2)}, V_{K_0}^{(4)}, \ldots, V_{K_0}^{(2q)}, \ldots, \\
&\ \left. V_\phi^{(4)}, V_\phi^{(6)}, \ldots, V_\phi^{(2r)}, \ldots \right\}.
\end{aligned}
$$

$V_J^{(m)}$, $m = 2$ or $m = 3, 5, 7, \ldots$ denotes the number of vertices in G with the insertion of one J and m ϕ's. $V_{K_0}^{(m)}$, $m = 2, 4, 6, \ldots$ stands for the number of vertices with the insertion of one K_0 and m ϕ's. Finally $V_\phi^{(m)}$, $m = 4, 6, 8, \ldots$ denotes the number of vertices with m ϕ's and neither J_μ nor K_0's.

Vertices with one K_0 do not contain derivatives. Vertices with one J_μ carry one momentum while vertices with only ϕ's carry two momenta.

In D dimensions the superficial degree of divergence for the graph G is

$$
d(G) = nD - 2I + \sum_k V_J^{(k)} + 2 \sum_j V_\phi^{(j)}. \tag{A.1}
$$

Use of the Euler's relation

$$
I = n + V - 1 \tag{A.2}
$$

with

$$
V = \sum_k V_J^{(k)} + \sum_j V_\phi^{(j)} + \sum_l V_{K_0}^{(l)} \tag{A.3}
$$

gives

$$d(G) = (D - 2)n + 2 - \sum_k V_J^{(k)} - 2 \sum_l V_{K_0}^{(l)} \tag{A.4}$$

The above formula shows that at a given loop order n the maximum superficial degree of divergence in the collection of graphs with N_J insertions of the composite operator F_μ^a, N_{K_0} insertions of the composite operator ϕ_0 and no ϕ's external legs is obtained when the number of vertices $V_J^{(k)}$ and $V_{K_0}^{(l)}$ is as small as possible.

This configuration is achieved by connecting all J_μ's and all K_0's along a chain of propagators and by inserting a sufficient number of additional propagators joining the above vertices in such a way to generate a n-loop graph. For that purpose one needs N_J vertices with one J_μ and N_{K_0} vertices with one K_0. There are $N_J + N_{K_0}$ lines in the external chain and $n - 1$ internal lines have to be added in order to get a n-loop graph.

The superficial degree of divergence is thus

$$d_{\max}(G) = Dn - (2(N_J + N_{K_0}) + 2(n - 1)) + N_J$$
$$= (D - 2)n + 2 - (N_J + 2N_{K_0}). \tag{A.5}$$

$d_{\max}(G) < 0$ if

$$N_J + 2N_{K_0} > (D - 2)n + 2. \tag{A.6}$$

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